Stabilizers and Simulating Entanglement

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Abstract

The boundary between classical information and quantum information is investigated. More specifically, the stabilizer formalism and the simulation of entangled states are considered. It is shown that Bell state correlations arising from measurements from the Pauli group can be simulated using local hidden variables. An explicit protocol for simulating GHZ state correlations using local hidden variables and two classical bits of communication is derived.

1 Introduction

Historically, fundamental physics was originally concerned with matter - what it was and how it moved. Later the emphasis switched to energy and how it was transformed and expressed. We know now that there can be no such thing as information without its physical representation, be it encoded as ink on a page, as 1's and 0's in a digital computer or as qubits in a quantum mechanical system. In short: information is physical. As more is learned it seems reasonable to wonder if physics is informational - if an information-theoretical framework is more fruitful in terms of generating new laws and principles.

Classical information theory is now seen to be part of the more general theory of quantum information. The building block of Q.I.T. is the qubit. Qubits have the states $|0\rangle$ and $|1\rangle$ corresponding to the classical 0 and 1 but they also have a continuum of states in between. We can write the state of the qubit as $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where α and β are complex numbers. However when we observe or measure the state of the qubit it only ever gives 0 or 1 probabilistically (0 with probability $|\alpha|^2$, 1 with probability $|\beta|^2$). Because probabilities must sum to one we have the constraint $|\alpha|^2 + |\beta|^2 = 1$. It is obvious that all the possible states of a qubit can be depicted as the surface of a unit sphere:

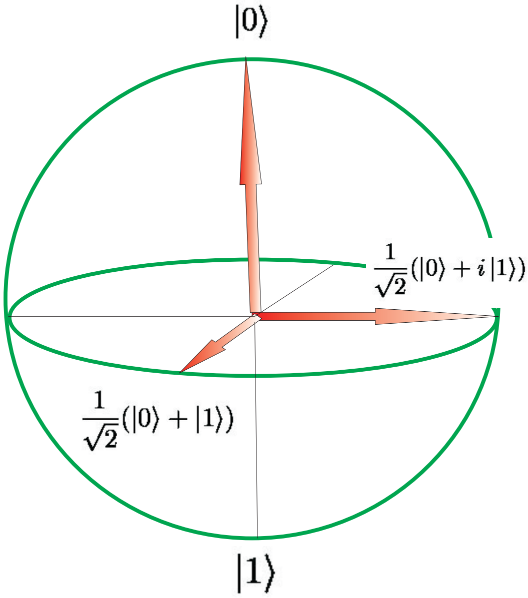


Figure 1: The Bloch Sphere

By convention $|0\rangle$ and $|1\rangle$ correspond to $+1$ and -1 of the z-axis; $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ correspond to $+1$ and -1 of the x-axis and $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ correspond to $+1$ and -1 of the y-axis.

When more than one qubit is involved the overall state is calculated by using the tensor product. For example; if $|\Psi\rangle_1 = \alpha_1|0\rangle + \beta_1|1\rangle$ and $|\Psi\rangle_2 = \alpha_2|0\rangle + \beta_2|1\rangle$ then the overall $|\Psi\rangle = |\Psi\rangle_1 \otimes |\Psi\rangle_2 = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle$. The resulting state $|\Psi\rangle$ is separable back into two one-qubit states. Because of the very many degrees of freedom in the quantum mechanical arena (Hilbert space) the majority of states are non-separable or entangled e.g.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

This is a Bell state. When the first qubit is measured it will return an answer of either 0 or 1 with equal probability and will also collapse the wave function to $|00\rangle$ or $|11\rangle$ respectively. A measurement of the second qubit always gives the same result as the measurement of the first qubit. In fact **the measurement correlations of a Bell state are stronger than could ever exist between classical systems**.

It is interesting to examine situations where classical and quantum information can be compared quantitatively. In the case of superdense coding two classical bits of information can be sent for the cost of measuring a (2-qubit) Bell state shared between Alice and Bob. Using teleportation one (possibly unknown) qubit can be sent from Alice to Bob for the cost of two classical bits of information and a shared (2-qubit) Bell state.

Results like Shor's factoring algorithm and the Deutsch-Jozsa algorithm clearly show that important and exploitable differences exist between quantum and classical information. The stabilizer formalism, however, and the Knill-Gottesmann theorem which results from it depict the surprisingly large extent to which quantum evolution and measurement can be *efficiently* simulated on a classical computer. This is the subject of Section 2.

It has been shown experimentally that *nature is not locally realistic* i.e. that either one or both of the classical assumptions of **locality** (a measurement here will not affect a measurement elsewhere) and **realism** (that physical properties have definite values independent of observation) are wrong. John Bell predicted this by showing that quantum correlations can violate an inequality which encapsulates the maximum allowable correlation by a locally realistic theory. The violation, therefore, of a Bell inequality indicates that something intrinsically quantum is going on. In order to quantify just how much, one can ask the question: " What classical resources are required to achieve the same correlations as this quantum state ? ". This question is answered for two specific cases in Section 3.

2 Stabilizer Formalism

The stabilizer formalism allows for an unusual but often very compact way of representing quantum states and processes.

Stabilizer : An operator O **stabilizes** the state $|\psi\rangle$ if $O|\psi\rangle = |\psi\rangle$.

Group theory: A group is a set G (with **elements** g_i) in addition to an operation $\bullet$ which together satisfy certain properies like **closure**.

The Pauli Group on one qubit is the set $\{\pm 1, \pm i, X, Y, Z\}$ combined with the operation Matrix Multiplication where

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$X^2 = Y^2 = Z^2 = I$$

$$Y = iXZ.$$

The Pauli group on n qubits is the set containing all the different possible tensor product combinations of n Pauli operators e.g for two qubits: $\{\pm 1, \pm i, X \otimes X, X \otimes Y, \ldots, Z \otimes Y, Z \otimes Z\}$.

The **Stabilizer** S is a **subgroup** of the Pauli Group which satisfies:
(a) The elements of S **commute** ...$g_1 g_2 = g_2 g_1$
(b) -I is **not** an element.

As an explicit example consider the Bell state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

which can also be described by the Stabilizer **generated by**

$$< X \otimes X, Z \otimes Z > .$$

There are other operators which stabilize this state (the identity, I, being an obvious example) but these can all be formed by multiplication using $X \otimes X$ and $Z \otimes Z$. Instead of writing out the state in its computational basis (e.g. $|0\rangle$ and $|1\rangle$), we can identify this state uniquely by listing the operators which stabilize it or, even more compactly, listing the generators of the operators which stabilize it

The **Clifford Group** contains the single-qubit Hadamard transform R:

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

the phase gate P:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and the controlled-NOT (CNOT) gate, also known as the XOR:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The Clifford group also contains other gates (e.g. Z) but these can be formed using just the three gates above ($P^2 = Z$). We say the Clifford group **is generated by** controlled-not, Hadamard and Phase gates.

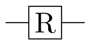
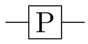
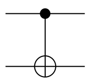
R	$X \to Z$ $Z \to X$	
P	$X \to Y$ $Z \to Z$	
CNOT	$X \otimes I \to X \otimes X$ $I \otimes X \to I \otimes X$ $Z \otimes I \to Z \otimes I$ $I \otimes Z \to Z \otimes Z$	

Table 1: Generators of the Clifford group

There are two different but equivalent ways of looking at how a quantum process occurs. One can consider the state $|\Psi\rangle$ being acted on by a unitary (satisfies $UU^\dagger = 1$) operator so that the state changes to $|\Psi\rangle'$. Alternatively one can consider the operators which have that state as an eigenvector being changed to new operators: operator M becomes $M' = UMU^\dagger$ (note that the new operator M' has $|\Psi\rangle'$ as an eigenvector). The latter viewpoint is adopted in the stabilizer formalism. Table 1 shows how the Pauli operators are changed (or conjugated) by the Hadamard, Phase and CNOT operators respectively.

Note: The Clifford group operators conjugate Pauli operators to Pauli operators.

To specify a state using its representation in the computational basis requires specifying 2^{n-1} coefficients where n is the number of qubits in the state. For example a 3-qubit state has 8 basis vectors: $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle$ and $|111\rangle$ with the normalization constraint specifying the 8^{th} coefficient. There is a result from group theory which states that the generating set for an n-qubit stabilizer state is of size n (recall that the generating set for the 2-qubit Bell state had 2 elements). Each generator takes $2n+1$ bits to specify; each Pauli requires two bits to specify and it takes one bit to specify a coefficient of ± 1 ($\pm i$ is excluded because iX, for example, multiplied by itself produces $-I$ in the stabilizer which is against the rules). Not only can the state be specified with $2n+1$ bits it can also be updated after a Clifford gate or Pauli measurement in $O(n^2)$ time. Using this result it is clear that a quantum computation (with these gate/measurement constraints) of m steps can be performed in $O(mn^2)$ time on a classical computer. This was the result found by Daniel Gottesman and Emanuel Knill.

Theorem 1 (Knill-Gottesman theorem) *Any quantum computer performing only: a) Clifford group gates, b) measurements of Pauli group operators, and c) Clifford group operations conditioned on classical bits, which may be the results of earlier measurements, can be perfectly simulated in polynomial time on a classical computer.*

It is surprising to note that these gates are sufficient to produce entanglement:

- Start with an initial state $|00\rangle$

- Apply the Hadamard gate:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- Now apply a CNOT:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This is the maximally entangled Bell state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Again, it can be described by the Stabilizer **generated by** $< X \otimes X, Z \otimes Z >$.

It is now clear that the presence of entanglement doesn't automatically bestow extra computational power. Amazingly, the Clifford gates and Pauli measurements are also sufficient to perform teleportation.

Two very important gates which lie outside the Clifford Group are the Toffoli gate and $\pi/8$ rotation of the Bloch sphere. One of these is **required for universal computation**. In fact, any other gate outside the Clifford group would do but these two are in a particularly convenient and useful form already.

3 Simulating Entanglement

If Alice and Bob share the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ then standard QM tells us:

If Alice performs a projective measurement on her qubit (**in the Z direction**) she will obtain the result $|0\rangle$ (spin-up) with probability $\frac{1}{2}$ or $|1\rangle$ (spin-down) also with probability $\frac{1}{2}$.

If Bob then performs a measurement **in the same basis** he will obtain either spin-up or spin down with 100% probability depending on Alice's result i.e. there is a total correlation between their results.

Using the normal convention for X,Y and Z measurement axes (see picture of Bloch sphere above) we can list the transformations from one basis to another (ignoring normalization factors):

$$\begin{aligned}
|+\rangle &\approx |0\rangle + |1\rangle \\
|-\rangle &\approx |0\rangle - |1\rangle \\
|0\rangle &\approx |+\rangle + |-\rangle \\
|1\rangle &\approx |+\rangle - |-\rangle \\
|p\rangle &\approx |0\rangle + i|1\rangle \\
|m\rangle &\approx |0\rangle - i|1\rangle \\
|0\rangle &\approx |p\rangle + |m\rangle \\
|1\rangle &\approx -i(|p\rangle - |m\rangle) \\
|p\rangle &\approx \alpha^*|+\rangle + \alpha|-\rangle \\
|m\rangle &\approx \alpha|+\rangle - \alpha^*|-\rangle \\
|+\rangle &\approx \alpha|p\rangle + \alpha^*|m\rangle \\
|-\rangle &\approx \alpha^*|p\rangle + \alpha|m\rangle
\end{aligned}$$

Note: $\{|0\rangle, |1\rangle\}$,$\{|p\rangle, |m\rangle\}$ and $\{|+\rangle, |-\rangle\}$ are the basis vectors for Z,Y and X respectively. Also, $\alpha = 1 - i$ and $\alpha^* = 1 + i$.

Because this Bell state can be rewritten in the X basis as

$$|\psi\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$$

an equivalent correlation occurs if Alice and then Bob both measure in the X basis ... either they both measure the state $|+\rangle$ or they both measure $|-\rangle$.

If the state is written in the Y basis then it is clear that if Alice and Bob both measure in the Y basis they will obtain opposite results i.e. results are completely anti-correlated:

$$|\psi\rangle = \frac{|pm\rangle + |mp\rangle}{\sqrt{2}}$$

However if Alice and Bob use different measurement bases e.g. X_{Alice} and Z_{Bob} (written XZ in the table below) or Z_{Alice} and X_{Bob} then their measurement outcomes are completely uncorrelated.

It is easy to check the correlations between Alice and Bob for all possible measurement choices (these correlations are derived explicitly in the appendix for the GHZ state).

Measurements	Alice's Result	Bob's result	% Correlation
XX	+1	+1	100%
XX	-1	-1	100%
XY	+1	±1	50%
XY	-1	±1	50%
XZ	+1	±1	50%
XZ	-1	±1	50%
YX	+1	±1	50%
YX	-1	±1	50%
YY	+1	-1	0%
YY	-1	+1	0%
YZ	+1	±1	50%
YZ	-1	±1	50%
ZX	+1	±1	50%
ZX	-1	±1	50%
ZY	+1	±1	50%
ZY	-1	±1	50%
ZZ	+1	+1	100%
ZZ	-1	+1	100%

But this state of affairs can be achieved by using three shared random bits of **classical information** and the following protocol:

- Label the random, independent bits (i.e. from $\{0,1\}$) b_1 , b_2 and b_3.

- If measuring along the X axis then output the the digit b_1.

- If measuring along the Y axis then Alice outputs the digit given by b_2 and Bob outputs the opposite, $\overline{b_2}$.

- If measuring along the Z axis then output the the digit b_3.

The rules of the game, then, are as follows:

The parties all have access to a random (arbitrarily long) string of bits. Obviously any two bits will be correlated, on average, 50% of the time. These bits can be thought of as local hidden variables. The parties also are given a list of instructions to follow (the protocol).

In general, if a number of parties share an entangled quantum state, quantum correlations are apparent in the joint probability distribution of the parties' measurement outcomes.

If this probability distribution can't be reproduced using local hidden variables (LHVs) (the shared bits of the above example) and a protocol then it is a purely quantum effect.

It is necessary, therefore, to use some classical communication (bits) between parties to reproduce the probability distribution.

In this paper measurement options have been restricted to X, Y or Z basis projective measurements. In general, a full analysis requires considering all measurements of the Von Neumann type. In that case LHV's would not suffice to reproduce Bell state correlations and 1 bit of classical communication is necessary.

In the literature there are two main models used to try to quantify the amount of classical communication required, the bounded (worst-case) communication model and the average communication model. Here the worst-case model will be considered.

The previous example above did not require *any* classical communication to simulate but others (e.g GHZ state) do.

$$GHZ = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

Einstein believed in **locality and realism**. The following argument (called the GHZ paradox) refutes local realism (i.e LHV models):

It is easy to check that the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is an eigenstate of the following operators (note that XYY is an abbreviation of $X \otimes Y \otimes Y$):

$$XYY|GHZ\rangle = -|GHZ\rangle \quad (1)$$
$$YXY|GHZ\rangle = -|GHZ\rangle \quad (2)$$
$$YYX|GHZ\rangle = -|GHZ\rangle \quad (3)$$
$$XXX|GHZ\rangle = |GHZ\rangle \quad (4)$$

Also note that

$$(XYY)(YXY)(YYX) = -(XXX)$$

Inspecting the first 3 eigenvalue equations (1-3) above implies that the spin along the x direction of one particle may be ascertained with certainty by measuring the y spin component of the other two particles (i.e. if the outcomes along y are *both* + 1 or *both* - 1 then the outcome along x must be -1 otherwise if the outcomes are + 1 *and* -1 then the outcome along x must be 1).

If one assumes that a measurement on one particle doesn't affect the other 2 particles (**locality**) then the **reality** implies that there exist 6 elements of physical reality corresponding to the measurement results (±1) of X and Y on each of the three qubits. For clarity we should label the qubits A,B and C and denote the elements of reality $m_x(A), m_x(B), m_x(C), m_y(A), m_y(B)$ and $m_y(C)$.

From (1-4) the elements must satisfy

$$\begin{aligned} m_x(A)m_y(B)m_y(C) &= -1 \\ m_y(A)m_x(B)m_y(C) &= -1 \\ m_y(A)m_y(B)m_x(C) &= -1 \\ m_x(A)m_x(B)m_x(C) &= 1 \end{aligned}$$

but this gives rise to a contradiction because

$$(m_x(A)m_y(B)m_y(C))(m_y(A)m_x(B)m_y(C))(m_y(A)m_y(B)m_x(C)) = m_x(A)m_x(B)m_x(C) = -1$$

4 An explicit protocol for simulating the GHZ state using two bits of classical communication

It is possible to analyze every possible set of outcomes arising from X,Y and Z measurements on the GHZ state (the measurement schemes resulting in correlations are worked out explicitly in the appendix). Listed in the table below are all the resultant correlations. These are only the measurement schemes that result in correlations. The majority of measurement schemes (e.g. YYZ) resulted in 50% correlatons between all parties (i.e. no meaningful correlation). These have been omitted.

Measurement Scheme	A's outcome	A-B Correlation	A-C Correlation	B-C Correlation
XXX	+1	50	50	**100**
XXX	−1	50	50	**0**
XYY	+1	50	50	**0**
XYY	−1	50	50	**100**
XZZ	+1	50	50	**100**
XZZ	−1	50	50	**100**
YXY	+1	50	50	**100**
YXY	−1	50	50	**0**
YYX	+1	50	50	**0**
YYX	−1	50	50	**100**
YZZ	+1	50	50	**100**
YZZ	−1	50	50	**100**
ZXZ	+1	50	**100**	50
ZXZ	−1	50	**100**	50
ZYZ	+1	50	**100**	50
ZYZ	−1	50	**100**	50
ZZX	+1	**100**	50	50
ZZX	−1	**100**	50	50
ZZY	+1	**100**	50	50
ZZY	−1	**100**	50	50
ZZZ	+1	**100**	**100**	**100**
ZZZ	−1	**100**	**100**	**100**

One protocol to simulate these correlations is as follows:

- If any of the parties measures along the Z axis, they output b_1.

- Alice sends Bob a bit which tells him whether she performed an X measurement or a Y measurement. If no bit is received then this indicates that Alice has performed a Z measurement. Regardless of whether she chose to do an X or Y measurement she outputs b_2.

- Bob performs his measurement and tells Charlie what to output. If Bob

measures in the X basis then he outputs b_3; if he measures in the Y basis he outputs $\overline{b_3}$ In the cases where Bob has chosen to do the same measurement as Alice he sends the bit 0 to Charlie which tells him:
"If you measure X output $(b_3 + b_2) mod 2$. If you measure Y output b_4".
In the cases where Bob decides to perform an X when A has measured Y or he performs a Y when A has measured X then he sends the bit 1 to Charlie which tells him:
" If you measure X output b_4. If you measure Y output $(b_3 + b_2) mod 2$." If no bit is sent (this happens when either Alice or Bob has decided to do a Z basis measurement) then Charlie outputs b_4 for either an X or Y measurement.

Two classical bits have been communicated in order to recreate the correlations of the GHZ state.

This protocol is somewhat convoluted and doubtless can be improved upon. The fact that Z measurements lead to *not* communicating a bit suggests that the average communication cost is lower than two bits.

Any state that can be arrived at by using gates from the Clifford Group and the initial state $|0\rangle^{\otimes n}$ is called a stabilizer state. The GHZ state and many other highly entangled states are stabilizer states. The stabilizer formalism allows for measurements in the X,Y or Z direction only. It is an interesting question, then, to consider the simulation of entangled states *in the context of the stabilizer formalism.* In that context it might be possible to formulate a generic protocol for the simulation of *any* stabilizer state.

5 Conclusion

Methods of quantifying the differences between quantum and classical information have been discussed. It has been shown what quantum processes and measurements can be efficiently simulated on a classical computer. Two protocols for simulating entangled states using local hidden variables have been derived; one of which requires classical communication between parties.

6 Acknowledgements

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7 Appendix: Explicit Calculation of GHZ correlations

Note: $|\Psi\rangle_3$ is a 3 qubit state; $|\gamma\rangle_2$ is a 2-qubit state etc. If Alice performs a measurement and obtains the outcome $+1$ (-1) then this is denoted $\oplus_A$ $(\ominus_A)$. Likewise when Bob performs a measurement we denote the outcome $\oplus_B$ or $\ominus_B$ (N.B. These symbols are purely to denote measurement outcomes and have no relation to the operation of tensor addition).

Take $|\Psi\rangle_3 = |+{+}{+}\rangle + |+{-}{-}\rangle + |-{+}{-}\rangle + |-{-}{+}\rangle$. If Alice measures in the X basis then this collapses the state to one of two possible states, depending on her outcome: If she gets $+1$ then the state collapses to $|\alpha\rangle_2 =$

$|++\rangle + |--\rangle$; if she gets -1 then it collapses to $|\beta\rangle_2 = |+-\rangle + |-+\rangle$. These new states will, in turn, be measured by Bob, again with possible outcomes ± 1 and again collapsing the wavefunction to one of four possible 1-qubit states: γ, δ, ϵ and ϕ.

To keep track of all possible sets of outcomes (and hence any possible correlations), the measurement schemes will be written out as follows:

Measurement Scheme: **XXX**

Initial State In X Basis:	$\vert\Psi\rangle_3 = \vert+++\rangle + \vert+--\rangle + \vert-+-\rangle + \vert--+\rangle$			
Alice Measures:	$\oplus_A \vert\alpha\rangle_2$		$\ominus_A \vert\beta\rangle_2$	
New State In X Basis	$\vert++\rangle + \vert--\rangle$		$\vert+-\rangle + \vert-+\rangle$	
Bob Measures:	$\oplus_A \oplus_B \vert\gamma\rangle$	$\oplus_A \ominus_B \vert\delta\rangle$	$\ominus_A \oplus_B \vert\epsilon\rangle$	$\ominus_A \ominus_B \vert\phi\rangle$
New State In X Basis	$\vert+\rangle$	$\vert-\rangle$	$\vert-\rangle$	$\vert+\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$100\%(\oplus_A), 0\%(\ominus_A)$

Using a grid like that above it is obvious from the state's position in the grid what the previous measurement results were (e.g a state arrived at by Alice obtaining a +1 measurement and collapsing the wavefunction will always be on the left hand side of the central dividing line). It can therefore be written simply as:

Measurement Scheme: **XXX**

Initial State In X Basis:	$\vert+++\rangle + \vert+--\rangle + \vert-+-\rangle + \vert--+\rangle$			
Alice Measures:	$\vert++\rangle + \vert--\rangle$		$\vert+-\rangle + \vert-+\rangle$	
New State In X Basis	$\vert++\rangle + \vert--\rangle$		$\vert+-\rangle + \vert-+\rangle$	
Bob Measures:	$\vert+\rangle$	$\vert-\rangle$	$\vert-\rangle$	$\vert+\rangle$
New State In X Basis	$\vert+\rangle$	$\vert-\rangle$	$\vert-\rangle$	$\vert+\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$100\%(\oplus_A), 0\%(\ominus_A)$

Measurement Scheme: **XYY**

Initial State In X Basis:	$\|+++\rangle + \|+--\rangle + \|-+-\rangle + \|--+\rangle$			
Alice Measures:	$\|++\rangle + \|--\rangle$		$\|+-\rangle + \|-+\rangle$	
New State In Y Basis	$\|pm\rangle + \|mp\rangle$		$\|pp\rangle + \|mm\rangle$	
Bob Measures:	$\|m\rangle$	$\|p\rangle$	$\|p\rangle$	$\|m\rangle$
New State In Y Basis	$\|m\rangle$	$\|p\rangle$	$\|p\rangle$	$\|m\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$0\%(\oplus_A), 100\%(\ominus_A)$

Measurement Scheme: **XZZ**

Initial State In X Basis:	$\|+++\rangle + \|+--\rangle + \|-+-\rangle + \|--+\rangle$			
Alice Measures:	$\|++\rangle + \|--\rangle$		$\|+-\rangle + \|-+\rangle$	
New State In Z Basis	$\|00\rangle + \|11\rangle$		$\|00\rangle - \|11\rangle$	
Bob Measures:	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
New State In Z Basis	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$100\%(\oplus_A), 100\%(\ominus_A)$

Measurement Scheme: **YXY**

Initial State In Y Basis:	$\alpha^*\|ppp\rangle + \alpha\|ppm\rangle + \alpha\|pmp\rangle + \alpha^*\|pmm\rangle$ $+\alpha\|mpp\rangle + \alpha^*\|mpm\rangle + \alpha^*\|mmp\rangle + \alpha\|mmm\rangle$			
Alice Measures:	$\alpha^*\|pp\rangle + \alpha\|pm\rangle$ $\alpha\|mp\rangle + \alpha^*\|mm\rangle$		$\alpha\|pp\rangle + \alpha^*\|pm\rangle$ $\alpha^*\|mp\rangle + \alpha\|mm\rangle$	
New State In X Basis	$\alpha^*\|++\rangle + \alpha\|+-\rangle$ $\alpha\|-+\rangle + \alpha^*\|--\rangle$		$\alpha\|++\rangle + \alpha^*\|+-\rangle$ $\alpha^*\|-+\rangle + \alpha\|--\rangle$	
Bob Measures:	$\alpha^*\|+\rangle + \alpha\|-\rangle$	$\alpha\|+\rangle + \alpha^*\|-\rangle$	$\alpha\|+\rangle + \alpha^*\|-\rangle$	$\alpha^*\|+\rangle + \alpha\|-\rangle$
New State In Y Basis	$\|p\rangle$	$\|m\rangle$	$\|m\rangle$	$\|p\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$100\%(\oplus_A), 0\%(\ominus_A)$

Measurement Scheme: **YYX**

Initial State In Y Basis:	$\alpha^*\|ppp\rangle + \alpha\|ppm\rangle + \alpha\|pmp\rangle + \alpha^*\|pmm\rangle$ $+\alpha\|mpp\rangle + \alpha^*\|mpm\rangle + \alpha^*\|mmp\rangle + \alpha\|mmm\rangle$			
Alice Measures:	$\alpha^*\|pp\rangle + \alpha\|pm\rangle$ $\alpha\|mp\rangle + \alpha^*\|mm\rangle$		$\alpha\|pp\rangle + \alpha^*\|pm\rangle$ $\alpha^*\|mp\rangle + \alpha\|mm\rangle$	
New State In Y Basis	$\alpha^*\|pp\rangle + \alpha\|pm\rangle$ $\alpha\|mp\rangle + \alpha^*\|mm\rangle$		$\alpha\|pp\rangle + \alpha^*\|pm\rangle$ $\alpha^*\|mp\rangle + \alpha\|mm\rangle$	
Bob Measures:	$\alpha^*\|p\rangle + \alpha\|m\rangle$	$\alpha\|p\rangle + \alpha^*\|m\rangle$	$\alpha\|+\rangle + \alpha^*\|m\rangle$	$\alpha^*\|p\rangle + \alpha\|m\rangle$
New State In X Basis	$\|-\rangle$	$\|+\rangle$	$\|+\rangle$	$\|-\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$0\%(\oplus_A), 100\%(\ominus_A)$

Measurement Scheme: **YZZ**

Initial State In Y Basis:	$\alpha^*\|ppp\rangle + \alpha\|ppm\rangle + \alpha\|pmp\rangle + \alpha^*\|pmm\rangle$ $+\alpha\|mpp\rangle + \alpha^*\|mpm\rangle + \alpha^*\|mmp\rangle + \alpha\|mmm\rangle$			
Alice Measures:	$\alpha^*\|pp\rangle + \alpha\|pm\rangle$ $\alpha\|mp\rangle + \alpha^*\|mm\rangle$		$\alpha\|pp\rangle + \alpha^*\|pm\rangle$ $\alpha^*\|mp\rangle + \alpha\|mm\rangle$	
New State In Z Basis	$\|00\rangle + i\|11\rangle$		$\|00\rangle - i\|11\rangle$	
Bob Measures:	$\alpha^*\|p\rangle + \alpha\|m\rangle$	$\alpha\|p\rangle + \alpha^*\|m\rangle$	$\alpha\|+\rangle\alpha^*\|m\rangle$	$\alpha^*\|p\rangle + \alpha\|m\rangle$
New State In Z Basis	$\|-\rangle$	$\|+\rangle$	$\|+\rangle$	$\|-\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	50%	$0\%(\oplus_A), 100\%(\ominus_A)$

Measurement Scheme: **ZXZ**

Initial State In Z Basis:	$\|000\rangle + \|111\rangle$			
Alice Measures:	$\|00\rangle$		$\|11\rangle$	
New State In X Basis	$\|++\rangle + \|+-\rangle$ $\|-+\rangle + \|--\rangle$		$\|++\rangle - \|--\rangle$ $-\|-+\rangle + \|--\rangle$	
Bob Measures:	$\|+\rangle + \|-\rangle$	$\|+\rangle + \|-\rangle$	$\|+\rangle - \|-\rangle$	$-\|+\rangle + \|-\rangle$
New State In Z Basis	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	$100\%(\oplus_A), 100\%(\ominus_A)$	50%

Measurement Scheme: **ZYZ**

Initial State In Z Basis:	$\|000\rangle + \|111\rangle$			
Alice Measures:	$\|00\rangle$		$\|11\rangle$	
New State In Y Basis	$\|pp\rangle + \|pm\rangle$ $\|mp\rangle + \|mm\rangle$		$-\|pp\rangle + \|pm\rangle$ $+\|mp\rangle - \|mm\rangle$	
Bob Measures:	$\|p\rangle + \|m\rangle$	$\|p\rangle + \|m\rangle$	$-\|p\rangle + \|m\rangle$	$\|p\rangle - \|m\rangle$
New State In Z Basis	$\|0\rangle$	$\|0\rangle$	$-i\|1\rangle$	$i\|1\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
50%	$100\%(\oplus_A), 100\%(\ominus_A)$	50%

Measurement Scheme: **ZZX**

Initial State In Z Basis:	$\|000\rangle + \|111\rangle$			
Alice Measures:	$\|00\rangle$		$\|11\rangle$	
New State In Z Basis	$\|00\rangle$		$\|11\rangle$	
Bob Measures:	$\|0\rangle$			$\|1\rangle$
New State In X Basis	$\|+\rangle + \|-\rangle$			$\|+\rangle - \|-\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
$100\%(\oplus_A), 100\%(\ominus_A)$	50%	50%

Measurement Scheme: **ZZY**

Initial State In Z Basis:	$\|000\rangle + \|111\rangle$			
Alice Measures:	$\|00\rangle$		$\|11\rangle$	
New State In Z Basis	$\|00\rangle$		$\|11\rangle$	
Bob Measures:	$\|0\rangle$			$\|1\rangle$
New State In Y Basis	$\|p\rangle + \|m\rangle$			$-i(\|p\rangle - \|m\rangle)$

A-B Correlations	A-C Correlations	B-C Correlations
$100\%(\oplus_A), 100\%(\ominus_A)$	50%	50%

Measurement Scheme: **ZZZ**

Initial State In Z Basis:	$\|000\rangle + \|111\rangle$			
Alice Measures:	$\|00\rangle$		$\|11\rangle$	
New State In Z Basis	$\|00\rangle$		$\|11\rangle$	
Bob Measures:	$\|0\rangle$			$\|1\rangle$
New State In Z Basis	$\|0\rangle$			$\|1\rangle$

A-B Correlations	A-C Correlations	B-C Correlations
$100\%(\oplus_A), 100\%(\ominus_A)$	$100\%(\oplus_A), 100\%(\ominus_A)$	$100\%(\oplus_A), 100\%(\ominus_A)$