**CNS185**

**Derivation[1] of dynamical equations describing networks of neurons.**

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When an action potential arrives at a synapse, neurotransmitter is released from that synapse, diffuses across the synaptic cleft, and temporarily binds to **ligand-gated channels** in the postsynaptic membrane. These are small pores in the membrane which open only when a neurotransmitter molecule is bound (ligated) to them. When they open, ions can travel across them, carrying current.

There are many channels in the membrane; we shall assume that a large number of them open almost instantaneously and that they then close at random, following a poisson distribution of opening times. Thus the conductivity associated with a particular type of channel, after an action potential arrived at time $t_a$, is modeled as

$$\sigma = \begin{cases} \sigma^0 \cdot e^{-(t-t_a)/\tau} & \text{if } t > t_a \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\tau$ is some characteristic time constant and $\sigma^0$ is a constant. The current which flows into the cell due to this $\sigma$ will be

$$i = \sigma(V_{driving} - V) \tag{2}$$

where $V$ is the cell's current membrane potential and $V_{driving}$ is the driving potential associated with the particular ion that the pore is selectively permeable to (see the Nernst equation in your notes from lecture). Since the threshold for firing the cell is not far from the resting potential of the cell, we shall take $V$ to be approximately constant[2], and take $\sigma^0(V_{driving} - V)$ (from equations (1) and (2)) to be a constant called $i^0$.

We shall add two indices to $i^0$, to indicate which synapse it corresponds to. Thus, $i^0_{c_1 c_2}$ indicates the constant associated with the synapse connecting cell 2 to cell 1. Also, we add an index to $t_a$ to indicate *which* cell fired an action potential at that time: thus, $t_a^{(c_2)}$ is an action potential firing time for cell 2. Furthermore, we want to account for the effects of *many* arriving action potentials, not just one; we model their effects on the conductivity (and hence current) as additive, so the total current into cell 1 due to action potentials from cell 2 is

$$i_{c_1 c_2}(t) = \sum_{t_a^{(c_2)} < t} i^0_{c_1 c_2} \cdot e^{-(t - t_a^{(c_2)})/\tau} \tag{3}$$

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[1]Note the assumptions and approximations in the text that lead to this derivation; different ones would have led to different final equations, naturally.

[2]The approximation that $V$ is constant is pretty good for Na pores, but is rather less good for K and Cl pores.

We now represent the firing in cell 2 as a series of delta functions,

$$f_2 = \sum_{t_a^{(c_2)}} \delta(t - t_a^{(c_2)}) \tag{4}$$

Integrated over a short period of time, this counts the number of action potentials that occurred in that period of time. Using this notation, we rewrite equation (3) as

$$i_{c_1 c_2}(t) = i^0_{c_1 c_2} \int_{-\infty}^{t} f_2(t') e^{-(t-t')/\tau} \mathrm{d}t' \tag{5}$$

Note that this is just a rewriting of equation (3)– we have done nothing fancy yet. Lets now change indices a little bit: instead of talking about cells 1 and 2, lets generalize slightly and talk about cells $j$ and $k$; and lets find the *total* current going into cell $j$. In other words, we will sum over all $k$ to find $i_{c_j}(t)$, the total input current to cell $j$ at time $t$.

$$i_{c_j} = \sum_k i^0_{c_j c_k} \int_{-\infty}^{t} f_k(t') e^{-(t-t')/\tau} \mathrm{d}t' \tag{6}$$

Having written it in this integral form allows us to differentiate this equation to find our dynamical equations:

$$\frac{\mathrm{d}i_{c_j}}{\mathrm{d}t} = \frac{-i_{c_j}}{\tau} + \sum_k i^0_{c_j c_k} f_k(t) \tag{7}$$

(to do this differentiation, note that $t$ appears twice, once inside the integral and once as one of the limits of the integral. This gives the two terms. Consult any calculus book if you're unclear about this. The derivation is very straightforward.)

Now consult your notes from lecture, and you will see that the firing rate $f_k(t)$ is arguably (i.e. under certain assumptions can be treated as) a smooth function of the input current to the cell:

$$f_k(t) = g(i_{c_k}) \tag{8}$$

(we implicitly used this smoothness assumption in the differentiation from (6) to (7).) Finally, lets rewrite all this to simplify: call $V_k = g(i_{c_k})$, $u_j = i_{c_j}$, $T_{jk} = \tau \cdot i^0_{c_j c_k}$. Then we get

$$\boxed{\tau \frac{\mathrm{d}u_j}{\mathrm{d}t} = -u_j + \sum_k T_{jk} V_k} \tag{9}$$

where, remember, $V_k = g(u_k)$.